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The β-space property in monotonically normal spaces and GO-spaces

Harold R. Bennett [a], David J. Lutzer [b,*]

[a] *Texas Tech University, Lubbock, TX 79409, USA*
[b] *College of William and Mary, Williamsburg, VA 23187-8795, USA*

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Abstract

In this paper we examine the role of the β-space property (equivalently of the MCM-property) in generalized ordered (GO-)spaces and, more generally, in monotonically normal spaces. We show that a GO-space is metrizable iff it is a β-space with a G_δ-diagonal and iff it is a quasi-developable β-space. That last assertion is a corollary of a general theorem that any β-space with a σ-point-finite base must be developable. We use a theorem of Balogh and Rudin to show that any monotonically normal space that is hereditarily monotonically countably metacompact (equivalently, hereditarily a β-space) must be hereditarily paracompact, and that any generalized ordered space that is perfect and hereditarily a β-space must be metrizable. We include an appendix on non-Archimedean spaces in which we prove various results announced without proof by Nyikos.

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* Corresponding author.
E-mail address: lutzer@math.wm.edu (D.J. Lutzer).

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1. Introduction

To say that a space (X,τ) is a β-space [14] means that there is a function g from $\{1,2,3,\ldots\}\times X$ to τ such that each $g(n,x)$ is a neighborhood of x and if $y\in g(n,x_n)$ for each n, then the sequence $\langle x_n\rangle$ has a cluster point in X. The function $g(n,x)$ in that definition is said to be a *β-function for X*. Many types of spaces have β-functions, e.g., semi-stratifiable spaces [8], wΔ-spaces, strict p-spaces, and countably compact spaces [12].

Recently the β-space property re-emerged in a completely different context, namely the study of monotone modifications of topological properties. The following definition appears in [11].

Definition 1.1. A topological space is *monotonically countably metacompact* (MCM) if for each decreasing sequence $D=\{D_n\colon n<\omega\}$ of closed sets with $\bigcap\{D_n\colon n<\omega\}=\emptyset$, there is a sequence $\{U(n,D)\colon n<\omega\}$ of open sets satisfying:

(a) for each $n<\omega$, $D_n\subseteq U(n,D)$;
(b) $\bigcap\{U(n,D)\colon n<\omega\}=\emptyset$;
(c) if $C=\{C_n\colon n<\omega\}$ is a decreasing sequence of closed sets with empty intersection, and if $C_n\subseteq D_n$ for each n, then $U(n,C)\subseteq U(n,D)$ for each n.

Note that the open sets $U(n,D)$ in that definition depend on an entire decreasing sequence $D=\langle D_n\rangle$ of closed sets with empty intersection. In a subsequent paper, Ying and Good [23] proved:

Lemma 1.2. *For a T_1-space X, the following are equivalent*:

(a) *X is MCM*;
(b) *for each point $x\in X$ there is a sequence $\{g(n,x)\colon n<\omega\}$ of open neighborhoods of x such that if $\{D_n\colon n<\omega\}$ is a decreasing sequence of closed sets with empty intersection, then the sets $G(n,D_n)=\bigcup\{g(n,x)\colon x\in D_n\}$ satisfy $\bigcap\{G(n,D_n)\colon n<\omega\}=\emptyset$*;
(c) *X is a β-space.*

In the light of Lemma 1.2 we will use the terms "β-space" and "MCM-space" interchangeably in this paper, depending upon which one sounds better in a given context.

It is well known that every GO-space is hereditarily normal and has the much stronger property called monotone normality [13]. By way of contrast, it is also well known that every GO-space is hereditarily countably metacompact, but familiar examples show that GO-spaces may fail to have the monotone countable metacompactness property. The following examples were announced in [11].

Example 1.3. Neither the Sorgenfrey line nor the Michael line nor the lexicographic product $\mathbb{Z}^{\omega_1}$ (a LOTS that is a topological group) is MCM (equivalently, none of the three is a β-space).

Those examples immediately lead one to ask which GO-spaces are MCM and what is the role of the MCM property (equivalently, the β-space property) among GO-spaces.

Section 2 contains metrization theorems for GO-spaces that involve the β-space property. The first provides another solution of the equation

$$\text{GO-space} + G_\delta\text{-diagonal} + (?) = \text{metrizable}.$$

We prove:

Theorem 1.4. *A GO-space is metrizable if and only if it is a β-space with a G_δ-diagonal.*

Theorem 1.5. *A GO-space is metrizable if and only if it is a quasi-developable β-space.*

Theorem 1.5 is a corollary of a general theorem that asserts:

Theorem 1.6. *Any regular β-space with a σ-point-finite base is developable.*

In Section 3 we investigate the hereditary β-space property. We begin by using a theorem of Balogh and Rudin [2] and a stationary set argument to show that:

Proposition 1.7. *Any monotonically normal space* (*and in particular, any GO-space*) *that is hereditarily a β-space is hereditarily paracompact.*

The rest of Section 3 is devoted to proving metrization theorems that depend on the hereditary β-property (equivalently, the hereditary MCM property). We first show that a GO-space with a σ-closed discrete set is metrizable if and only if each of its subspaces is a β-space and then investigate what happens if "X has a σ-closed discrete dense set" is weakened to "X is perfect". Normally, one expects that metrization theorems for GO-spaces with σ-closed-discrete dense sets will not generalize to perfect spaces because normally one runs into Souslin space problems when one considers perfect GO-spaces that do not, a priori, have σ-closed-discrete dense sets. We get around this problem using results of Qiao and Tall, coupled with some results about non-Archimedean spaces that were announced many years ago by Peter Nyikos. Based on those results, we prove:

Theorem 1.8. *A GO-space is metrizable if and only if it is perfect and each of its subspaces is a β-space.*

Because the required results of Nyikos have never been published, we include our proofs of them in Section 4 of this paper.

Recall that a *generalized ordered space* (GO-space) is a triple $(X, <, \tau)$ where $<$ is a linear ordering of X and τ is a Hausdorff topology on X that has a base of order-convex subsets (possibly including singletons). Probably the best-known GO-spaces are the Sorgenfrey line and the Michael line. If τ is the usual open interval topology of the ordering, then $(X, <, \tau)$ is a *linearly ordered topological space* (LOTS). Čech proved that the GO-spaces are exactly those spaces that embed topologically in some LOTS [5].

In this paper we reserve the symbols $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ for the sets of all integers, rational, and real numbers, respectively. For any ordinal x, $\mathrm{cf}(x)$ denotes the cofinality of x. We will need to distinguish between subsets of X that are *relatively discrete* (i.e., are discrete when topologized as subspaces of X) and sets that are both closed and discrete subsets of X. We will also need to distinguish between dense sets that are σ-relatively discrete subsets of X (i.e., that are unions of countably many relatively discrete subsets of X) and those that are σ-closed-discrete (i.e., countable unions of closed discrete subsets of X).

2. Metrization and the β-space property

In this section, we investigate how the β-space property interacts with other topological properties to provide metrization theorems. Recall that any LOTS with a G_δ-diagonal is metrizable [15] while GO-spaces with G_δ-diagonals may fail to be metrizable (e.g., the Sorgenfrey and Michael lines). The β-space property is exactly what is missing, and we have:

Proposition 2.1. *A GO-space is metrizable if and only if it is a β-space with a G_δ-diagonal.*

Proof. Half of the proposition is trivial. To prove the other half, recall that a space X is paracompact if it is a GO-space with a G_δ-diagonal [15]. Also recall that any space X is semi-stratifiable if it is a β-space with a G_δ^*-diagonal (see Theorem 7.8(ii) of [12]) and that any paracompact space with a G_δ-diagonal has a G_δ^*-diagonal. Therefore X is semi-stratifiable. Hence X is metrizable [15]. $\square$

Our next result is a general theorem—it is not restricted to GO-spaces.

Proposition 2.2. *A T_3, β-space with a σ-point-finite base is developable. A T_3 space is metrizable if and only if it is a collectionwise normal β-space with a σ-point-finite base.*

Proof. The second assertion of the proposition follows from the first because any collectionwise normal developable space is metrizable.

The first assertion is already known: Hodel [14] noted that any space with a σ-point-finite base is a γ-space and proved (in his Proposition 4.2) that any T_1-space that is both a γ-space and a β-space must be developable. An alternate approach begins with $\bigcup\{\mathcal{B}(n)\colon n \geqslant 1\}$, a σ-point-finite base for X. We may modify that base if necessary so that $\mathcal{B}(2k)$ is the collection of all singleton isolated points of X for each $k \geqslant 1$. We may also assume that each $\mathcal{B}(n)$ is closed under finite intersections so that, if $x \in \bigcup \mathcal{B}(i)$ then there is a member $B(i,x) \in \mathcal{B}(i)$ that is the smallest member of $\mathcal{B}(i)$ that contains x. Now let $g(n,x)$ be a β-function for X. Because X is first-countable, we may assume that $\{g(n,x)\colon n \geqslant 1\}$ is a decreasing local base at x for each point $x \in X$ and that if x is isolated, then $g(n,x) = \{x\}$ for each n. One first proves that for any fixed $x \in X$ and $n \geqslant 1$, there is some $m \geqslant n$ with $x \in \bigcup \mathcal{B}(m)$ and $B(m,x) \subseteq g(n,x)$, where $B(m,x)$ is the smallest member of $\mathcal{B}(m)$ that contains x. Then for each fixed x and n we may define $\phi(n,x)$ to be the first integer $m \geqslant n$ having $x \in \bigcup \mathcal{B}(m)$ and

$B(m,x) \subseteq g(n,x)$. Observe that for each fixed x, $\phi(n,x) \leqslant \phi(x,n+1)$. Now define $h(n,x) = \bigcap\{B(i,x)\colon\ x \in \bigcup \mathcal{B}(i)$ and $i \leqslant \phi(x,n)\}$. Then $h(n+1,x) \subseteq h(n,x)$ and $h(n,x) \subseteq g(n,x)$ so that h is also a β-function for X and $\{h(n,x)\colon\ n \geqslant 1\}$ is a local base at x for each point of X. Verify that if $p \in h(n,x_n)$ for each $n \geqslant 1$, then $\langle x_n \rangle$ clusters to p. Now apply a theorem of Aull [1] to show that X, having a σ-point-finite base, must be quasi-developable. To complete the proof, all we need to show is that X is perfect. For any closed set C, let $G_n = \bigcup\{h(n,x)\colon\ x \in C\}$. Then G_n is an open set and $\bigcap\{G_n\colon\ n \geqslant 1\} = C$. □

We do not know whether the previous proposition can be generalized to quasi-developable spaces. (That would be a generalization, because Aull has proved that any space with a σ-point-finite base is quasi-developable.) A recent paper [16] claimed that any quasi-developable β-space must be developable, but some details of the proof are unclear.

Whether or not each quasi-developable β-space is developable, we have the following equivalence for GO-spaces:

Corollary 2.3. *A GO-space is metrizable if and only if it is a quasi-developable β-space.*

Proof. To prove the non-trivial half of the corollary, suppose X is a GO-space that is quasi-developable and a β-space. Then by [3,15] X has a σ-point-finite base and is collectionwise normal. Now apply Proposition 2.2. □

3. The hereditary β-space property

In our paper [6] we proved that any GO-space that is hereditarily a β-space must be hereditarily paracompact. The key to the proof was a pressing down lemma argument that showed:

Lemma 3.1. *No stationary subset of a regular uncountable cardinal can be hereditarily a β-space in its relative topology.*

In [6], we then combined Lemma 3.1 with a characterization of paracompactness in generalized ordered spaces from [9] to get the desired result. Since the time of that earlier paper, Balogh and Rudin [2] have significantly generalized the result from [9], showing that a monotonically normal space fails to be paracompact if and only if it contains a closed subspace that is homeomorphic to a stationary set in a regular uncountable cardinal. Combining that result with Lemma 3.1 gives:

Corollary 3.2. *A monotonically normal space that is hereditarily a β-space is hereditarily paracompact.*

In the remainder of this section we prove that the hereditary MCM property is a natural component of metrizability in GO-spaces. We begin by recalling the following lemma [10,4].

Lemma 3.3. *Suppose X is a GO-space with a dense subset that is σ-closed-discrete. Then*:

(a) *X is perfect* (*i.e., each closed set is a G_δ-set*) *and first-countable.*
(b) *There is a sequence $\{\mathcal{H}(n)\colon n \geqslant 1\}$ of open covers of X such that for each $p \in X$, $\bigcap\{St(p, \mathcal{H}(n))\colon n \geqslant 1\}$ has at most two points.*
(c) (Faber's Metrization Theorem [10]) *The GO-space X is metrizable if and only if the sets $R = \{x \in X\colon [x, \to) \in \tau\}$, $L = \{x \in X\colon (\leftarrow, x] \in \tau\}$ and $I = \{x \in X\colon \{x\} \in \tau\}$ are each σ-closed-discrete in X.*

Theorem 3.4. *Let $(X, <, \tau)$ be a GO-space. Then X is metrizable if and only if X has a σ-closed-discrete dense set and is hereditarily a β-space.*

Proof. Any metric space has a σ-closed-discrete dense set and is hereditarily a β-space. To prove the converse, suppose X has a σ-closed-discrete dense subset E and is hereditarily a β-space. We will apply Faber's metrization theorem in part (c) of Lemma 3.3. The set of isolated points, being a subset of E, is σ-closed-discrete. We prove that the set $R = \{x \in X\colon [x, \to) \in \tau\}$ is σ-closed-discrete; the proof for the set $L = \{x \in X\colon (\leftarrow, x] \in \tau\}$ in Faber's theorem (see Lemma 3.3(c)) is analogous.

By Lemma 3.3, X is perfect so that each relatively discrete subset of X is σ-closed-discrete. Therefore, it will be enough to show that the set R is the union of countably many relatively discrete, but perhaps not closed, subsets. To that end, for each $x \in R$, find a sequence $\{g(n, x)\colon n \geqslant 1\}$ of sets that satisfy Lemma 1.2 for the subspace (R, τ_R). Replacing those sets by smaller sets if necessary, we may assume:

(a) $\{g(n, x)\colon n \geqslant 1\}$ is a decreasing local base at x in the subspace (R, τ_R);
(b) each set $g(n, x)$ is contained in some member of the cover $\mathcal{H}(n)$ described in Lemma 3.3(b);
(c) $g(n, x) \subseteq [x, \to)$ for each $x \in R$ and each n;
(d) if $a < b < c$ are points of R with $a, c \in g(n, x)$ then $b \in g(n, x)$.

Let $\mathcal{G}(n) = \{g(n, x)\colon x \in R\}$ and define $R(n) = \{x \in R\colon St(x, \mathcal{G}(n)) \subseteq [x, \to)\}$. Then $R(n) \subseteq R(n + 1)$ for each n. Let $R^* = \bigcup\{R(n)\colon n \geqslant 1\}$. For each $x \in R^*$ there is some n with $x \in R(n)$. Then $St(x, \mathcal{G}(k)) \subseteq [x, \to)$ for each $k \geqslant n$. But then $g(k, x)$ is the unique member of $\mathcal{G}(k)$ that contains x for each $k \geqslant n$. For suppose $y \in R$ and $x \in g(k, y)$ where $k \geqslant n$. Then $y \in g(k, y) \subseteq St(x, \mathcal{G}(k)) \subseteq [x, \to)$ yields $x \leqslant y$. But if $x < y$, then $x \in g(k, y)$ makes $g(k, y) \subseteq [y, \to)$ impossible, contrary to (c) in the description of how the sets $g(k, z)$ are chosen for $z \in R$. Therefore the sets $St(x, \mathcal{G}(k)) = g(k, x)$ form a neighborhood base for x in the space (R, τ_R) so that the subspace (R^*, τ_{R^*}) is developable and hence is metrizable. Applying Faber's metrization theorem, we see that the set R^* is the countable union of subspaces that are relatively discrete. We claim $R = R^*$. If not, then there is a point $y \in R - R^*$. Then $y \notin R(n)$ for each n so there must be some points $x_n \in R$ with $y \in g(n, x_n) \not\subseteq [y, \to)$. Because $g(n, x_n) \subseteq [x_n, \to)$ we must have $x_n < y$.

There cannot be an infinite sequence $n_1 < n_2 < \cdots$ with $x_{n_1} = x_{n_2} = \cdots$ because then $y \in g(n_k, x_{n_k}) = g(n_k, x_{n_1})$ would make it impossible for the sets $g(n, x_{n_1})$ to be a decreasing local base at the point x_{n_1}. We claim that there cannot be a sequence $m_1 < m_2 < \cdots$

with $x_{m_1} > x_{m_2} > \cdots$. For suppose such a decreasing subsequence exists. Then for each $j \geqslant m_2$ we have

$$\{x_{m_2}, x_{m_1}, y\} \subseteq [x_{m_j}, y] \cap R \subseteq g(m_j, x_{m_j}) \subseteq g(j, x_{m_j}) \in \mathcal{G}(j)$$

which shows that

$$\{x_{m_2}, x_{m_1}, y\} \subseteq St\big(y, \mathcal{G}(j)\big) \subseteq St\big(y, \mathcal{H}(j)\big)$$

for each $j \geqslant 2$ and that is impossible in the light of the special properties of the covers $\mathcal{H}(n)$ described in part (b) of Lemma 3.3.

Therefore the sequence $\langle x_n \rangle$ has no constant subsequences and no strictly decreasing subsequences, so there must be a strictly increasing subsequence $x_{n_1} < x_{n_2} < \cdots$. Let $A_k = \{x_{n_i}\colon i \geqslant k\}$ and observe that A_k has no limit points in R. Hence $\{A_k\colon k \geqslant 1\}$ is a decreasing sequence of closed sets with empty intersection. However, with $G(k, A_k)$ defined as in Lemma 1.2, we have $y \in \bigcap\{G(k, A_k)\colon k \geqslant 1\}$ and that is impossible. Hence $R = R^*$, so R is the union of countably many subspaces, each being relatively discrete. The same is true of the subset L and we may now apply Faber's metrization theorem to complete the proof. □

Experience has shown that many results proved for GO-spaces having σ-closed-discrete dense sets become axiom-sensitive when stated for the broader class of perfect GO-spaces. It is somewhat surprising that Theorem 3.4 is not of this type. We begin with a result about dense metrizable subspaces. Then, by combining Theorem 3.4 with some known results about non-Archimedean spaces (i.e., spaces with a base that is a tree under reverse inclusion) we obtain a new metrization theorem for perfect GO-spaces.

Proposition 3.5. *Let X be a first-countable GO-space that is hereditarily a β-space. Then X has a dense metrizable subspace.*

Proof. We need two results from the literature.

(a) Any first-countable GO-space contains a dense non-Archimedean subspace, i.e., a dense subspace having a base of open, convex sets that is a tree under reverse inclusion.
(b) Any first-countable, non-Archimedean β-space is metrizable.

The first is due to Qiao and Tall [20] (who proved the result for first-countable LOTS, but a slight modification of their proof establishes the result for first-countable GO-spaces). The second is due to Nyikos [17]. No proof of the second result has appeared in print and we include a proof and relevant definitions in the final section of this paper.

Now suppose X is a first countable GO-space. Let Y be a dense non-Archimedean subspace of X. If X is hereditarily a β-space, then Y is a β-space. Now apply assertion (b) above to show that Y is metrizable. □

Theorem 3.6. *Suppose X is a GO-space. Then X is metrizable if and only if X is perfect and hereditarily MCM.*

Proof. To prove the non-trivial part of the theorem, suppose that X is perfect and hereditarily MCM. Apply Proposition 3.5 to find a dense metrizable subspace Y of X. Then Y contains a dense subset D that is the union of countably many subsets $D(n)$, each being relatively discrete. But then, X being perfect, each $D(n)$ is the union of countably many subsets $D(n,k)$ where each $D(n,k)$ is a closed discrete subspace of X. Now apply Theorem 3.4 to conclude that X is metrizable. $\square$

As noted in the Introduction, any compact or countably compact space is β-space because every sequence in a countably compact space has a cluster point. Hence the lexicographic square is a β-space, as is the ordinal space $[0,\omega_1)$. However, the *hereditary* β-space property is another matter, and we have the following question.

Question 3.7. Is there a compact, first-countable LOTS X that is hereditarily a β-space and not metrizable?

Note that, in the light of Corollary 3.5, if X is a first-countable compact LOTS that is hereditarily a β-space, then X has a dense metrizable subspace, as does each subspace of X. Also note that by Theorem 3.6 and assertion (b) in the proof of Theorem 3.6, many kinds of subspaces of such an X will be metrizable. These include perfect subspaces (a class that includes all separable subspaces and, more generally, all subspaces with a σ-closed discrete dense subset), non-Archimedean subspaces, and subspaces with a point-countable base (because, according to a result of Chaber [7] (see also Theorem 7.9 of [12]) any first-countable, paracompact β-space with a point-countable base must be metrizable). Other results in the literature suggest that one place to look for the required example is in the branch spaces of certain trees [21,22].

4. Appendix on non-Archimedean spaces

A regular space X is *non-Archimedean* if it has a base that is a tree under reverse inclusion. Basic topological results about such spaces were announced by Nyikos in [17–19] but, Nyikos has informed us, no proof of the one result needed in this paper (namely that a first-countable non-Archimedean β-space is metrizable) has ever been published. The goal of this appendix is to provide the required proof. Our approach is as follows. First we will show that any non-Archimedean space is paracompact. Next we will show that any first-countable non-Archimedean β-space is developable and then, from general metrization theory, we will conclude that any first-countable, non-Archimedean β-space is metrizable. It happens to be true that any non-Archimedean space is a GO-space, but we will not use that fact in our proofs.

Lemma 4.1. *Suppose $\mathcal{B}$ is a tree-base for the non-Archimedean space X. Then*:

(a) *each member of $\mathcal{B}$ is clopen*;
(b) *each subspace of X is ultraparacompact*;

(c) *if $p \in X$ and if $\mathcal{C} \subseteq \mathcal{B}$ has $p \in \bigcap \mathcal{C}$, then either $\bigcap \mathcal{C}$ is a neighborhood of p or else $\bigcap \mathcal{C} = \{p\}$ and $\mathcal{C}$ is a neighborhood base at p.*

Proof. For any $p \in X$, let $\mathcal{B}(p) = \{B \in \mathcal{B}\colon\ p \in B\}$. Then $\mathcal{B}(p)$ is well-ordered by reverse inclusion and if $p \in B_1 \cap B_2$ (where $B_i \in \mathcal{B}$) then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

To prove (a), let $B \in \mathcal{B}$. Let p be any limit point of B and suppose $p \notin B$. Choose $q \in B$. Because $p \neq q$, we may choose $B' \in \mathcal{B}$ with $p \in B' \subseteq X - \{q\}$. Then $B \cap B' \neq \emptyset$ and $B \subseteq B'$ is impossible, so that $p \in B' \subseteq B$, contrary to $p \notin B$. Hence B is clopen.

To prove (b), recall that a space Y is *ultraparacompact* if each open cover of Y has a pairwise disjoint open refinement. Let $\mathcal{U}$ be any collection of open subsets of X. Let $\mathcal{D}$ be the collection of all $B \in \mathcal{B}$ that are contained in some member of $\mathcal{U}$. Let $\mathcal{V}$ be the collection of all minimal members of $\mathcal{D}$ with respect to the tree ordering (i.e., reverse inclusion) of $\mathcal{B}$. Then $\mathcal{V}$ refines $\mathcal{U}$, is pairwise disjoint, and has $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. It follows that every open subspace of X, and hence every subspace of X, is ultraparacompact.

To prove (c), suppose $\mathcal{C} \subseteq \mathcal{B}$ and $p \in \bigcap \mathcal{C}$ and suppose that $\bigcap \mathcal{C}$ is not a neighborhood of p. We claim that $\bigcap \mathcal{C} = \{p\}$. For suppose there are at least two points p, q in $\bigcap \mathcal{C}$ and choose any member $B_0 \in \mathcal{B}$ with $p \in B_0 \subseteq X - \{q\}$. Then B_0 meets each $C \in \mathcal{C}$ and B_0 cannot contain any member of $\mathcal{C}$. Hence B_0 is a subset of each member of $\mathcal{C}$ and therefore $B_0 \subseteq \bigcap \mathcal{C}$. But that makes $\bigcap \mathcal{C}$ a neighborhood of p which is impossible. Hence $\bigcap \mathcal{C} = \{p\}$. Let B_1 be any member of $\mathcal{B}$ that contains p. Because $\bigcap \mathcal{C}$ is not a neighborhood of p, we must have $B_1 \not\subseteq \bigcap \mathcal{C}$ so that for some $C \in \mathcal{C}$, $B_1 \not\subseteq C$. Hence $C \subseteq B_1$ as required to show that $\mathcal{C}$ is a local base at p. □

Proposition 4.2. *If X is a non-Archimedean β-space in which points are G_δ-sets, then X is metrizable.*

Proof. Part (c) of Lemma 4.1 shows that a non-Archimedean space in which points are G_δ-sets must be first-countable.

Let $\mathcal{B}$ be a tree-base for the space X and let $g(n, x)$ be a β-function for X as described in Section 1. Because we can replace each $g(n, x)$ by a smaller neighborhood of x and still have a β-function, we may assume that $g(n, x) \in \mathcal{B}$ and that $\{g(n, x)\colon\ n \geqslant 1\}$ is a local base at x. We may also assume that $g(n + 1, x)$ is a proper subset of $g(n, x)$ unless x is isolated and that $g(n, x) = \{x\}$ for each n if x is isolated.

We now describe a partition process that will be applied to various sets $g(n, x)$. If x is isolated, then $g(n, x) = g(n + 1, x) = \{x\}$ and we let $\mathcal{W}(g(n, x)) = \{g(n + 1, x)\}$. If x is not isolated, then the set $S = g(n, x) - g(n + 1, x)$ is not empty and, by part (a) of Lemma 4.1, S is open. Let the members of $\mathcal{W}(g(n, x))$ be $g(n + 1, x)$ together with all members of the collection $\{g(k, y)\colon\ k \geqslant n + 1 \text{ and } g(k, y) \subseteq S\}$ that are minimal in the ordering of the tree $(\mathcal{B}, \supseteq)$. Then $\mathcal{W}(g(n, x))$ is a pairwise disjoint open cover of $g(n, x)$ by sets of the form $g(k, y) \in \mathcal{B}$ where $k \geqslant n + 1$. Note that if $g(k, y) \in \mathcal{W}(g(n, x))$ with $y \neq x$, then $x \notin g(k, y)$. For each set $W \in \mathcal{W}(g(n, x))$ choose one point $y(W) \in W$ such that $W = g(k, y(W))$ and $k \geqslant n + 1$, making sure that if $W = g(n + 1, x)$, then $y(W) = x$. Let $C(g(n, x)) = \{y(W)\colon\ W \in \mathcal{W}(g(n, x))\}$. Let $\mathcal{H}(0) = \{X\}$. Given $\mathcal{H}(n)$ for some n, define

$$\mathcal{H}(n + 1) = \bigcup \bigl\{\mathcal{W}\bigl(g(m, x)\bigr)\colon\ g(m, x) \in \mathcal{H}(n)\bigr\}.$$

Let $C(n) = \{y(W)\colon W \in \mathcal{H}(n)\}$. Each $\mathcal{H}(n)$ is a pairwise disjoint cover of X by members of $\mathcal{B}$ that have the form $g(y, k)$ for exactly one $y \in C(n)$ and $k \geqslant n$.

We claim that the sequence $\mathcal{H}(1), \mathcal{H}(2), \ldots$ is a development for X. Fix any $p \in X$. Then p belongs to exactly one member of $\mathcal{H}(n)$ so that $St(p, \mathcal{H}(n))$ is a member of $\mathcal{H}(n)$ and has the form $g(k_n, y_n)$ where $y_n \in C(n)$ and $k_n \geqslant n$. Furthermore, $g(k_{n+1}, y_{n+1}) \subseteq g(k_n, y_n)$ because of the way that the collections $\mathcal{H}(n)$ were constructed. Because $k_n \geqslant n$ we have $p \in g(n, y_n)$ and therefore the sequence $y_1, y_2, \ldots$ must cluster at some point $q \in X$. Because $g(k_{n+1}, y_{n+1}) \subseteq g(k_n, y_n)$ we see that each $g(k_n, y_n)$ contains $\{y_m\colon m \geqslant n\}$ and therefore the point q is a point of the closure of each $g(k_n, y_n)$. But $g(k_n, y_n)$ is clopen, being a member of $\mathcal{B}$, so that $\{p, q\} \subseteq \bigcap\{g(k_n, y_n)\colon n \geqslant 1\}$.

If infinitely many terms in sequence $y_1, y_2, \ldots$ are the same, say $y_n = y_N$ for each n in the infinite set I, then because $k_n \geqslant n$ the sets $g(k_n, y_n)$ form a local base at y_N so that $p, q \in \bigcap\{g(k_n, y_n)\colon n \geqslant 1\} = \{y_N\}$ forces $p = q = y_N$ and hence $\{St(p, \mathcal{H}(n))\colon n \geqslant 1\}$ is a local base at $\{p\}$.

If the sequence $y_1, y_2, \ldots$ has no constant subsequences, then there is a subsequence of distinct terms. For notational simplicity, assume that $y_i \neq y_j$ whenever $i \neq j$. Then we know that $y_n \notin g(k_{n+1}, y_{n+1})$ so that the set $T = \bigcap\{g(k_n, y_n)\colon n \geqslant 1\}$ contains no point y_k. Hence T cannot be a neighborhood of q even though $q \in T$. But by part (c) of Lemma 4.1 we know that since T is not a neighborhood of q, it must be true that $T = \{q\}$ and $\{g(k_n, y_n)\colon n \geqslant 1\}$ is a neighborhood base at q. But $\{p, q\} \subseteq T$ then forces $p = q$ so that, once again, $\{St(p, \mathcal{H}(n))\colon n \geqslant 1\}$ is a local base at p.

At this stage of the proof, we know that X is developable and paracompact (by part (b) of Lemma 4.1) and therefore metrizable. □

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